J. Math. Anal. Appl. 374 (2011) 442–457

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Half Dirichlet problem for matrix functions on the unit ball in Hermitian Clifford analysis ☆

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ARTICLE INFO

Article history:
Received 8 January 2010
Available online 13 August 2010
Submitted by T. Ransford

Keywords:
Hermitian Clifford analysis
Cauchy transformation
Dirichlet problem
Matrix functions

ABSTRACT

The simultaneous null solutions of the two complex Hermitian Dirac operators are focused on in Hermitian Clifford analysis, where the Hermitian Cauchy integral was constructed and will play an important role in the framework of circulant (2×2) matrix functions. Under this setting we will present the half Dirichlet problem for circulant (2×2) matrix functions on the unit ball of even dimensional Euclidean space. We will give the unique solution to it merely by using the Hermitian Cauchy transformation, get the solution to the Dirichlet problem on the unit ball for circulant (2×2) matrix functions and the solution to the classical Dirichlet problem as the special case, derive a decomposition of the Poisson kernel for matrix Laplace operator, and further obtain the decomposition theorems of solution space to the Dirichlet problem for circulant (2×2) matrix functions.

1. Introduction

Orthogonal Clifford analysis is a higher dimensional function theory offering both a generation of complex analysis in the plane and a refinement of classical harmonic analysis. The theory is centered around the concept of monogenic functions, i.e. null solutions of the first order vector-valued rotation invariant differential operator, called Dirac operator (see e.g. [1–4,14,18–21,27]). Under this setting, the half Dirichlet problems on the unit ball, on the upper half space and on Lipschitz surface of higher Euclidean space were considered by Delanghe, Sommen, Tao Qian, Mitrea and so on (see [15–17]). In [15,16], using the methods of orthogonal Clifford analysis, the solutions to the well-known Dirichlet problems (see e.g. [24,25]) were also gotten.

More recently, offering yet a refinement of the orthogonal case, Hermitian Clifford analysis in references e.g. [6–13] emerged as a new and successful branch of Clifford analysis. It focuses on the simultaneous null solutions of the two complex Hermitian Dirac operators, which are invariant under the action of the unitary group and were first studied in references e.g. [6,7]. The Cauchy integral formula for Hermitian monogenic functions defined in even dimensional Euclidean space taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ was constructed in the framework of circulant (2×2) matrix functions, and at the same time the intimate relationship with holomorphic function theory of several complex variables (see references e.g. [23,22]) was established by Brackx, De Schepper, Sommen and so on (see [11]). The new Hilbert-like matrix operator was revealed by the non-tangential boundary limits of Hermitian Cauchy type integral and analogues of characteristic properties of the matrix Hilbert transform in classical analysis and orthogonal Clifford analysis were given in [10]. The Hermitian Cauchy type integral and the related decomposition problems of continuous functions were discussed in [12,13].

☆ This research was supported in part by the National Natural Science Foundation of China under Grant Nos. 10871150, 60873249, 863 Project of China under Grant No. 2008AA01Z419, Post-Doctor Foundation of China under Grant No. 20090460316.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.08.015

Much recent progress can be also seen elsewhere. Under this setting it is natural to think of the half Dirichlet problem. However up to the present, as far as we know, it has not been studied. In this paper, based on [10–13,15–17], we will consider the half Dirichlet problem for circulant (2×2) matrix functions on the unit ball in the setting of Hermitian Clifford analysis. We will obtain the unique solution to it, and get the solution to the Dirichlet problem for circulant (2×2) matrix functions on the unit ball and the solution to the well-known Dirichlet problem by means of Hermitian Clifford analysis. As the special cases of it, we also give the solutions to the Dirichlet problems for analytic functions of one complex variable and for holomorphic functions of several complex variables respectively. Furthermore, we will derive a decomposition of the Poisson kernel of matrix Laplace operator, and obtain the decomposition theorems of solution space to the Dirichlet problem for circulant (2×2) matrix functions.

The paper is organized as follows. In Section 2, we recall some basic facts about Clifford algebras and Hermitian Clifford analysis which will be needed in the sequel. In Section 3, we will present and study the half Dirichlet problem for circulant (2×2) matrix functions on the unit ball, obtain the unique solution to it by merely using the Hermitian Cauchy transformation, and get the solution to the Dirichlet problem for circulant (2×2) matrix functions on the unit ball and the solution to the well-known Dirichlet problem by means of Hermitian Clifford analysis. As the special case, we also give the solution to the classical Dirichlet problem for analytic functions of one complex variable and for holomorphic functions of several complex variables respectively. In the last section we will derive a decomposition of the Poisson kernel of matrix Laplace operator, and obtain the decomposition theorems of solution space to the Dirichlet problems for circulant (2×2) matrix functions.

2. Preliminaries and notations

In this section we recall some basic facts about Clifford algebras and Hermitian Clifford analysis which will be needed in the sequel. More details can be also seen in references e.g. [1–5,14,18–21,27,6–13].

Let $\{e_1, e_2, \dots, e_m\}$ be an orthogonal basis of the Euclidean space $\mathbb{R}^m$, let $\mathbb{R}^m$ be endowed with a non-degenerate quadratic form of signature $(0, m)$ and let $\mathbb{R}_{0,m}$ be the 2^n-dimensional real Clifford algebra constructed over $\mathbb{R}^m$ with basis

$$\left\{e_{\mathcal{A}}\colon \mathcal{A} = \{h_1, \dots, h_r\} \in \mathcal{PN},\ 1 \leqslant h_1 < h_r \leqslant m\right\},$$

where $\mathcal{N}$ stands for the set $\{1, 2, \dots, m\}$ and $\mathcal{PN}$ denotes the family of all order-preserving subsets of $\mathcal{N}$. We denote $e_{\emptyset}$ as e_0 and $e_{\mathcal{A}}$ as $e_{h_1 \cdots h_r}$ for $\mathcal{A} = \{h_1, \dots, h_r\} \in \mathcal{PN}$. The product in $\mathbb{R}_{0,m}$ is defined by

$$\begin{cases} e_{\mathcal{A}} e_{\mathcal{B}} = (-1)^{N(\mathcal{A} \cap \mathcal{B})} (-1)^{P(\mathcal{A},\mathcal{B})} e_{\mathcal{A} \triangle \mathcal{B}}, & \text{if } \mathcal{A}, \mathcal{B} \in \mathcal{PN}, \\ \lambda\mu = \displaystyle\sum_{\mathcal{A},\mathcal{B} \in \mathcal{PN}} \lambda_{\mathcal{A}} \mu_{\mathcal{B}} e_{\mathcal{A}} e_{\mathcal{B}}, & \text{if } \lambda = \displaystyle\sum_{\mathcal{A} \in \mathcal{PN}} \lambda_{\mathcal{A}} e_{\mathcal{A}},\ \mu = \displaystyle\sum_{\mathcal{B} \in \mathcal{PN}} \mu_{\mathcal{B}} e_{\mathcal{B}}, \end{cases}$$

where $N(\mathcal{A})$ is the cardinal number of the set $\mathcal{A}$, and $P(\mathcal{A}, \mathcal{B}) = \sum_{j \in \mathcal{B}} P(\mathcal{A}, j)$, with $P(\mathcal{A}, j) = N(\mathcal{Z})$ and $\mathcal{Z} = \{i\colon\ i \in \mathcal{A},\ i > j\}$. It follows e_0 is the identity element, now written as 1 and that in particular

$$\begin{cases} e_i^2 = -1, & \text{if } i = 1, 2, \dots, m, \\ e_i e_j + e_j e_i = 0, & \text{if } 1 \leqslant i < j \leqslant m, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } 1 \leqslant h_1 < h_2 < \cdots < h_r \leqslant m. \end{cases}$$

Thus the real Clifford algebra $\mathbb{R}_{0,m}$ is a real linear, associative, but non-commutative algebra.

Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_{N(\mathcal{A})=k} a_{\mathcal{A}} e_{\mathcal{A}}$, $a_{\mathcal{A}} \in \mathbb{R}$, or still as $a = \sum_{N(\mathcal{A})=k} [a]_k$, where $[a]_k = \sum_{N(\mathcal{A})=k} e_{\mathcal{A}} [a]_{\mathcal{A}}$ is the so-called k-vector part of a $(k = 0, 1, 2, \dots, m)$. The Euclidean space $\mathbb{R}^m$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, x_2, \dots, x_m)$ with the Clifford vector $\underline{X}$ given by

$$\underline{X} = \sum_{j=1}^{m} e_j x_j.$$

The conjugation in $\mathbb{R}_{0,m}$ is defined as follows:

$$\bar{a} = \sum_{\mathcal{A}} a_{\mathcal{A}} \bar{e}_{\mathcal{A}}, \qquad \bar{e}_{\mathcal{A}} = (-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}, \qquad N(\mathcal{A}) = k, \quad a_{\mathcal{A}} \in \mathbb{R},$$

and hence

$$\overline{ab} = \bar{b}\bar{a}, \quad \text{for arbitrary } a, b \in \mathbb{R}_{0,m}.$$

Note that the square of a vector $\underline{X}$ is scalar-valued and equals the norm squared up to a minus sign $\underline{X}^2 = -\langle \underline{X}, \underline{X} \rangle = -|\underline{X}|^2$. The dual of the vector $\underline{X}$ is the vector-valued first order differential operator

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

which is called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart or holomorphy in the complex plane. As the Dirac operator factorizes the Laplacian, $\Delta_m = -\partial_{\underline{X}}^2$, monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the orthogonal case, since the fundamental group leaving the Dirac operator $\partial_{\underline{X}}$ invariant is the special orthogonal group $\mathrm{SO}(m;\mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is called a rotation invariant operator.

When allowing for complex constants and moreover taking the dimension to be even, say $m = 2n$, the same set of generators as above, $\{e_1, e_2, \ldots, e_{2n}\}$, still satisfying the above defining relation, may in fact also product the complex Clifford algebra $\mathbb{C}_{2n}$. As $\mathbb{C}_{2n}$ is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\mathbb{R}_{0,2n}$, any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, leading to the Hermitian conjugation $\lambda^\dagger = (a+ib)^\dagger = \bar{a} - i\bar{b}$, where the bar denotes the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti-involution for which $\bar{e}_j = -e_j$, $j = 1, 2, \ldots, 2n$. This Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_{\mathcal{A}} |\lambda_{\mathcal{A}}|^2)^{\frac{1}{2}}$. The above framework will be referred to as the Hermitian Clifford analysis, as opposed to the traditional orthogonal Clifford one. Hermitian Clifford analysis then focuses on the simultaneous null solutions of two Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, introduced as follows.

One of the ways for introducing Hermitian Clifford analysis is by considering the complex Clifford algebra $\mathbb{C}_{2n}$ and a so-called complex structure on it, i.e. an $\mathrm{SO}(2n,\mathbb{R})$-element J for which $J^2 = -\mathbf{1}$ (see e.g. [6–9]). More specifically, J is chosen to act upon the generators $e_1, e_2, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \quad \text{and} \quad J[e_{n+j}] = e_j, \quad j = 1, 2, \ldots, n.$$

Let us recall that the main objects of the Hermitian setting are then conceptually obtained by considering the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ and letting them act on the corresponding protagonists of the orthogonal framework. First of all, the so-called Witt basis elements $\{f_j, f_j^\dagger \mid j = 1, 2, \ldots, n\}$ for the complex Clifford algebra $\mathbb{C}_{2n}$ are obtained through the action of $\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements e_j:

$$f_j = \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - ie_{n+j}), \quad j = 1, 2, \ldots, n,$$
$$f_j^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + ie_{n+j}), \quad j = 1, 2, \ldots, n.$$

These Witt basis elements satisfy the Grassmann identities

$$f_j f_k + f_k f_j = f_j^\dagger f_k^\dagger + f_k^\dagger f_j^\dagger = 0, \quad j, k = 1, 2, \ldots, n,$$

and the duality identities

$$f_j f_k^\dagger + f_k^\dagger f_j = \delta_{jk}, \quad j, k = 1, 2, \ldots, n.$$

Next we identify a vector $\underline{X} = (X_1, X_2, \ldots, X_{2n}) = (x_1, x_2, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ with the Clifford vector $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$ and we denote by $\underline{X}|$ the action of the complex structure J on $\underline{X}$, i.e.

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^n (e_j y_j - e_{n+j} x_j).$$

Note that the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product, which implies that the Clifford vectors $\underline{X}$ and $\underline{X}|$ are anti-commutative. The Hermitian Clifford variables $\underline{Z}$ and $\underline{Z}^\dagger$ then arise through the action of the projection operators on the standard Clifford vector $\underline{X}$:

$$\underline{Z} = \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \frac{1}{2}(\underline{X} + i\underline{X}|),$$
$$\underline{Z}^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = -\frac{1}{2}(\underline{X} - i\underline{X}|).$$

They can be rewritten in terms of the Witt basis elements as

$$\underline{Z} = \sum_{j=1}^n f_j z_j \quad \text{and} \quad \underline{Z}^\dagger = (\underline{Z})^\dagger = \sum_{j=1}^n f_j^\dagger z_j^c,$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, 2, \ldots, n$. Finally, the Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are derived out of the orthogonal Dirac operator $\partial_{\underline{X}}$:

$$\partial_{\underline{Z}^\dagger} = \frac{1}{4}(\mathbf{1} + iJ)[\partial_{\underline{X}}] = \frac{1}{4}(\partial_{\underline{X}} + i\partial_{\underline{X}|}),$$
$$\partial_{\underline{Z}} = -\frac{1}{4}(\mathbf{1} - iJ)[\partial_{\underline{X}}] = -\frac{1}{4}(\partial_{\underline{X}} - i\partial_{\underline{X}|}),$$

where we have introduced

$$\partial_{\underline{X}|} = J[\partial_{\underline{X}}] = \sum_{j=1}^{n}(e_j\partial_{y_j} - e_{n+j}\partial_{x_j}).$$

In terms of the Witt basis elements, the Hermitian Dirac operators are expressed as

$$\partial_{\underline{Z}} = \sum_{j=1}^{n} f_j^\dagger \partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}^\dagger} = (\partial_{\underline{Z}})^\dagger = \sum_{j=1}^{n} f_j \partial_{z_j^c},$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j-planes, $j = 1, 2, \ldots, n$.

The Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are invariant under the action of a realization, denoted $\widetilde{U}(n)$, of the unitary group in terms of the Clifford algebras (see e.g. [6,7]). The group $\widetilde{U}(n) \subset \mathrm{Spin}(2n)$ is given by

$$\widetilde{U}(n) = \left\{ s \in \mathrm{Spin}(2n) \;\middle|\; \exists\theta \geqslant 0\colon\ \tilde{s}I = e^{-i\theta}I \right\}$$

its definition involving the self-adjoint primitive idempotent $I = I_1 I_2 \cdots I_n$, with $I_j = f_j f_j^\dagger = \frac{1}{2}(1 - ie_j e_{n+j})$, $j = 1, 2, \ldots, n$.

Finally observe for further use that the Hermitian vector variables and Dirac operators are isotropic, i.e.

$$(\underline{Z})^2 = \left(\underline{Z}^\dagger\right)^2 = 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0.$$

Whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}})$$

and one also has that

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\underline{Z} = |\underline{Z}|^2 = \left|\underline{Z}^\dagger\right|^2 = |\underline{X}|^2 = |\underline{X}|\,|^2.$$

For further use, we introduce the Hermitian oriented surface elements $d\sigma_{\underline{Z}}$ and $d\sigma_{\underline{Z}^\dagger}$ as follows

$$\varepsilon(\underline{Z}) = \frac{2}{w_{2n}}\frac{\underline{Z}}{|\underline{Z}|^{2n}} \quad \text{and} \quad \varepsilon^\dagger(\underline{Z}) = \frac{2}{w_{2n}}\frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}},$$
$$d\sigma_{\underline{Z}} = \sum_{j=1}^{n} f_j^\dagger\,\widehat{dz_j} \quad \text{and} \quad d\sigma_{\underline{Z}^\dagger} = \sum_{j=1}^{n} f_j\,\widehat{dz_j^c}.$$

Explicitly,

$$d\sigma_{\underline{Z}} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n(d\sigma_{\underline{X}} - id\sigma_{\underline{X}|}),$$
$$d\sigma_{\underline{Z}^\dagger} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n(d\sigma_{\underline{X}} + id\sigma_{\underline{X}|}),$$
$$\varepsilon = -(E + iE|), \qquad \varepsilon^\dagger = (E - iE|),$$

where $d\sigma_{\underline{X}}$ denotes the vector-valued oriented surface element and $d\sigma_{\underline{X}|} = J[d\sigma_{\underline{X}}]$. They are explicitly given by means of the following differential forms of order $2n-1$

$$d\sigma_{\underline{X}} = \sum_{j=1}^{n}\left(e_j(-1)^{j-1}\widehat{dx_j} + e_{n+j}(-1)^{n+j-1}\widehat{dy_j}\right),$$
$$d\sigma_{\underline{X}|} = \sum_{j=1}^{n}\left(e_j(-1)^{n+j-1}\widehat{dy_j} + e_{n+j}(-1)^{j}\widehat{dx_j}\right),$$

with

$$\widehat{dx_j} = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n,$$
$$\widehat{dy_j} = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n.$$

In this context the functions taking $\mathbb{C}_{2n}$-valued defined on an open subregion Ω of $\mathbb{R}^{2n}$ will be considered. The continuity, continuously differentiability, $\mathbb{L}_p$ $(1 < p < +\infty)$-integral and so on of the function $f = \sum_{\mathcal{A}} f_{\mathcal{A}} e_{\mathcal{A}} : \Omega(\subset \mathbb{R}^{2n}) \to \mathbb{C}_{2n}$ with $f_{\mathcal{A}} : \Omega(\subset \mathbb{R}^{2n}) \to \mathbb{C}$, the space of which are denoted respectively by $\mathscr{C}(\Omega, \mathbb{C}_{2n})$, $\mathscr{C}^1(\Omega, \mathbb{C}_{2n})$, $\mathbf{L}_p(\Omega, \mathbb{C}_{2n})$ and so on, are ascribed to each component $f_{\mathcal{A}}$ which is respectively continuous, continuously differential, $\mathbb{L}_p$-integrable and so on. A function $f(\underline{X})$ defined and differentiable in an open subregion Ω of $\mathbb{R}^{2n}$ with its boundary $\partial\Omega$ and taking values in $\mathbb{C}_{2n}$ is called (left) monogenic in Ω if $\partial_{\underline{X}} f(\underline{X}) = 0$.

Definition 2.1. The function $f(\underline{X}) : \partial\Omega(\subset \mathbb{R}^{2n}) \to \mathbb{C}_{2n}$ is said to be Hölder continuous if and only if there are constants $M > 0$ and $0 < \mu \leqslant 1$ satisfying for arbitrary $\underline{X}, \underline{Y} \in \partial\Omega$,

$$\left| f(\underline{X}) - f(\underline{Y}) \right| \leqslant M |\underline{X} - \underline{Y}|^{\mu}.$$

We denote the set of all Hölder continuous functions on $\partial\Omega$ as $\mathbb{H}^{\mu}(\partial\Omega, \mathbb{C}_{2n})$.

We introduce the particular circulant (2×2) matrices

$$\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^{\dagger}} \\ \partial_{\underline{Z}^{\dagger}} & \partial_{\underline{Z}} \end{pmatrix}, \qquad (\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})})^{\dagger} = \begin{pmatrix} \partial_{\underline{Z}^{\dagger}} & \partial_{\underline{Z}} \\ \partial_{\underline{Z}} & \partial_{\underline{Z}^{\dagger}} \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} \varepsilon & \varepsilon^{\dagger} \\ \varepsilon^{\dagger} & \varepsilon \end{pmatrix} \quad \text{and} \quad \underline{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix},$$

then $\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})} \mathcal{E} = \underline{\delta}(\underline{Z})$, i.e. $\mathcal{E}$ is the fundamental solution of $\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})}$ (see e.g. [6–9]).

In the same setting of circulant (2×2) matrices, we consider the functions $L_1, L_2, L \in \mathscr{C}^1(\Omega, \mathbb{C}_{2n})$ and the corresponding circulant (2×2) matrix functions in the following

$$\mathcal{L}_2^1 = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} \quad \text{and} \quad \mathcal{L}_0 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.$$

In the following context all operations of matrices such as addition and multiplication keep to the operation rules of usual numerical matrices.

Definition 2.2. Suppose that $\mathcal{L}_2^1(\mathcal{L}_0) \in \mathscr{C}^1(\Omega, \mathbb{C}_{2n})$ which means that each entry of $\mathcal{L}_2^1(\mathcal{L}_0)$ belongs to $\mathscr{C}^1(\Omega, \mathbb{C}_{2n})$. $\mathcal{L}_2^1(\mathcal{L}_0)$ is called as (left) H-monogenic if and only if it satisfies the following system

$$\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})} \mathcal{L}_2^1 = \mathbf{0} \quad (\mathbf{D}_{(\underline{Z}, \underline{Z}^{\dagger})} \mathcal{L}_0 = \mathbf{0}),$$

where $\mathbf{0}$ denotes the matrix with zero entries. Similarly, it is obvious in the following that $\mathcal{L}_2^1(\mathcal{L}_0) \in \mathscr{C}(\partial\Omega, \mathbb{C}_{2n})$, $\mathbb{H}^{\mu}(\partial\Omega, \mathbb{C}_{2n}), \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ $(1 < p < +\infty)$ and so on which mean each entry of $\mathcal{L}_2^1(\mathcal{L}_0)$ belongs to $\mathscr{C}(\partial\Omega, \mathbb{C}_{2n})$, $\mathbb{H}^{\mu}(\partial\Omega, \mathbb{C}_{2n})$, $\mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ and so on.

In the following we introduce

$$\underline{V} = \frac{1}{2}(\underline{Y} + i\underline{Y}|), \qquad \underline{V}^{\dagger} = -\frac{1}{2}(\underline{Y} - i\underline{Y}|),$$
$$dV_{(\underline{Z}, \underline{Z}^{\dagger})} = \left(dz_1 \wedge dz_1^c\right) \wedge \left(dz_2 \wedge dz_2^c\right) \wedge \cdots \wedge \left(dz_n \wedge dz_n^c\right),$$

where $dV_{(\underline{Z}, \underline{Z}^{\dagger})}$ denotes the Hermitian volume element.

For the functions $L_i \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ $(1 < p < +\infty,\ i = 1, 2)$, we define the orthogonal Cauchy type integrals as follows

$$\mathcal{C}[L_i](\underline{Y}) \triangleq (\mathcal{C}_{\partial\Omega} L_i)(\underline{Y}) = \int_{\partial\Omega} E(\underline{X} - \underline{Y})\, d\sigma_{\underline{X}} L_i(\underline{X}), \quad \underline{Y} \notin \partial\Omega,$$

$$\mathcal{C}|[L_i](\underline{Y}) \triangleq (\mathcal{C}|_{\partial\Omega} L_i)(\underline{Y}) = \int_{\partial\Omega} E|(\underline{X} - \underline{Y})\, d\sigma_{\underline{X}|} L_i(\underline{X}), \quad \underline{Y} \notin \partial\Omega,$$

which are well defined (see references e.g. [3]), where $E(\underline{X}) = \frac{1}{w_{2n}} \frac{\overline{\underline{X}}}{|\underline{X}|^{2n}}$, $E|(\underline{X}) = \frac{1}{w_{2n}} \frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}}$ and $d\sigma_{\underline{X}}$, $d\sigma_{\underline{X}|}$ as above. Then for $\underline{Y} \notin \partial\Omega$, $\partial_{\underline{Y}} \mathcal{C}[L_i](\underline{Y}) = 0$, $\partial_{\underline{Y}|} \mathcal{C}|[L_i](\underline{Y}) = 0$ $(i = 1, 2)$.

For the functions $\mathcal{L}_2^1, \mathcal{L}_0 \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$, the Hermitian Cauchy type integrals are defined by

$$\big[\mathbf{C}\mathcal{L}_2^1\big](\underline{Y}) = \int_{\partial\Omega} \mathcal{E}(\underline{Z} - \underline{V})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, \mathcal{L}_2^1(\underline{X}), \quad \underline{Y} \notin \partial\Omega, \tag{1}$$

$$[\mathbf{C}\mathcal{L}_0](\underline{Y}) = \int_{\partial\Omega} \mathcal{E}(\underline{Z} - \underline{V})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, \mathcal{L}_0(\underline{X}), \quad \underline{Y} \notin \partial\Omega, \tag{2}$$

where

$$d\Sigma_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix} \quad \text{with } d\sigma_{\underline{Z}} \text{ and } d\sigma_{\underline{Z}^\dagger} \text{ as above.}$$

Lemma 2.1. *Suppose that Ω is a $2n$-dimensional compact differentiable and oriented manifold contained in some open subset of $\mathbb{R}^{2n}$ with smooth boundary $\partial\Omega$. The functions $[\mathbf{C}\mathcal{L}_2^1](\underline{X})$ and $[\mathbf{C}\mathcal{L}_0](\underline{X})$ are defined similarly to $[\mathbf{C}\mathcal{L}_2^1](\underline{Y})$ and $[\mathbf{C}\mathcal{L}_0](\underline{Y})$ as above. If the functions $\mathcal{L}_2^1(\underline{X}), \mathcal{L}_0(\underline{X}) \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ $(1 < p < +\infty)$, then*

$$\text{(i)} \quad \textit{for arbitrary } \underline{X} \in \mathbb{R}^{2n} \setminus \partial\Omega, \quad \mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X}) = \mathbf{0}, \qquad \mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_0(\underline{X}) = \mathbf{0},$$
$$\textit{i.e.} \quad \mathcal{L}_2^1(\underline{X}),\ \mathcal{L}_0(\underline{X}) \textit{ are both H-monogenic},$$

$$\text{(ii)} \quad \big[\mathbf{C}\mathcal{L}_2^1\big]^{\pm}(\underline{T}) \triangleq \lim_{\Omega^{\pm}\ni\underline{X}\to\underline{T}} \big[\mathbf{C}\mathcal{L}_2^1\big](\underline{X}) = (-1)^{\frac{n(n+1)}{2}}(2i)^n \frac{1}{2}\big(\pm\mathcal{L}_2^1(\underline{T}) + \big[\mathbf{H}\mathcal{L}_2^1\big](\underline{T})\big), \quad \underline{T} \in \partial\Omega,$$

$$[\mathbf{C}\mathcal{L}_0]^{\pm}(\underline{T}) \triangleq \lim_{\Omega^{\pm}\ni\underline{X}\to\underline{T}} [\mathbf{C}\mathcal{L}_0](\underline{X}) = (-1)^{\frac{n(n+1)}{2}}(2i)^n \frac{1}{2}\big(\pm\mathcal{L}_0(\underline{T}) + [\mathbf{H}\mathcal{L}_0](\underline{T})\big), \quad \underline{T} \in \partial\Omega,$$

$$\text{(iii)} \quad \big[\mathbf{C}\mathcal{L}_2^1\big]^{\pm}(\underline{T}) \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n}) \quad \big([\mathbf{C}\mathcal{L}_0]^{\pm}(\underline{T}) \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})\big),$$

where the limits of (ii) *mean the non-tangential limits and it is the same in this context,*

$$\mathbf{H} = \frac{1}{2}\begin{pmatrix} \mathcal{H} + \mathcal{H}| & -\mathcal{H} + \mathcal{H}| \\ -\mathcal{H} + \mathcal{H}| & \mathcal{H} + \mathcal{H}| \end{pmatrix},$$

and

$$[\mathcal{H}f](\underline{T}) = \text{p.v.}\,2\int_{\partial\Omega} E(\underline{Y} - \underline{T})\, d\sigma_{\underline{Y}}\, f(\underline{Y}), \quad \underline{T} \in \partial\Omega,$$

$$[\mathcal{H}|f](\underline{T}) = \text{p.v.}\,2\int_{\partial\Omega} E|(\underline{Y} - \underline{T})\, d\sigma_{\underline{Y}|}\, f(\underline{Y}), \quad \underline{T} \in \partial\Omega,$$

which are both Cauchy principle value integrals in the sense of $\mathbb{L}_p$ $(1 < p < +\infty)$. When the variables are omitted without confusion and ambiguity, for convenience $[\mathcal{H}f](\underline{T})$, $[\mathcal{H}|f](\underline{T})$ are for short of $\mathcal{H}f$, $\mathcal{H}|f$ respectively and it is similar in the following context.

Proof. For $p = 2$, the results were also obtained in reference [10]. For $1 < p < +\infty$, $p \neq 2$, following the same argument contained in references e.g. [10,11], the results of Lemma 2.1 still hold.

In fact (i), since $\partial_{\underline{X}}\mathcal{C}[L_i](\underline{X}) = 0$, $\partial_{\underline{X}|}\mathcal{C}|[L_i](\underline{X}) = 0$ $(i = 1, 2)$ for arbitrary $\underline{X} \in \mathbb{R}^{2n} \setminus \partial\Omega$, where $\mathcal{C}[L_i](\underline{X})$, $\mathcal{C}|[L_i](\underline{X})$ are defined similarly to $\mathcal{C}[L_i](\underline{Y})$, $\mathcal{C}|[L_i](\underline{Y})$ as above, then

$$\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X}) = \mathbf{0}, \qquad \mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_0(\underline{X}) = \mathbf{0}, \quad \text{i.e.} \quad \mathcal{L}_2^1(\underline{X}),\ \mathcal{L}_0(\underline{X}) \text{ are both H-monogenic.}$$

(ii) From references e.g. [30,29,28], the boundedness of the Hilbert transforms $[\mathcal{H}f](\underline{T})$ and $[\mathcal{H}|f](\underline{T})$ in $\mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ follows, which derive the corresponding Plemelj–Sokhotzki formulas for $f(\underline{X}) \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$ (also see e.g. [29,16]). The results of (ii), (iii) of Lemma 2.1 follow. □

Remark 2.1. The cases $p = 1$ and $p = +\infty$ require more delicate analysis that will be considered in the forthcoming paper.

Remark 2.2. For the functions $[\mathbf{C}\mathcal{L}_2^1](\underline{X}), [\mathbf{C}\mathcal{L}_0](\underline{X}) \in \mathbb{H}^{\mu}(\partial\Omega, \mathbb{C}_{2n})$, the related results analogous to Lemma 2.1 can be also seen in references e.g. [12,13].

3. Half Dirichlet problem on the unit ball

In this section we will present and consider the half Dirichlet problem for circulant (2×2) matrix functions on the unit ball in Hermitian Clifford analysis, get the unique solution to it, and give the solution to the Dirichlet problem for circulant (2×2) matrix functions and the solution to the well-known Dirichlet problem by the way of Hermitian Clifford analysis. As the special cases, the solutions to the classical Dirichlet problems for analytic functions of one complex variable and for holomorphic functions of several complex variables can be given respectively by the way of Hermitian Clifford analysis.

In the sequel we denote the open unit ball centered at the origin by $B(1)$ whose closure is $\overline{B}(1)$, its boundary by S^{2n-1} and the open ball centered at the origin with the radius r $(r > 0)$ by $B(r)$. We introduce the following functions

$$\alpha(\underline{X}) = \frac{1}{2}(1 + i\underline{X}), \qquad \beta(\underline{X}) = \frac{1}{2}(1 - i\underline{X}), \quad \underline{X} \in \mathbb{R}^{2n},$$
$$\alpha|(\underline{X}) = \frac{1}{2}(1 + i\underline{X}|), \qquad \beta|(\underline{X}) = \frac{1}{2}(1 - i\underline{X}|), \quad \underline{X} \in \mathbb{R}^{2n}.$$

By directly calculating, it is easy to obtain the lemma as follows.

Lemma 3.1. *Suppose the functions $\alpha(\underline{X})$, $\alpha|(\underline{X})$ and $\beta(\underline{X})$, $\beta|(\underline{X})$ as above. Then*

$$\text{(i)} \quad \alpha(\underline{X})\beta(\underline{X}) = \alpha|(\underline{X})\beta|(\underline{X}) = \frac{1 - |\underline{X}|^2}{4}, \tag{3}$$
$$\text{(ii)} \quad \alpha(\underline{X}) = \alpha^{\dagger}(\underline{X}), \qquad \beta(\underline{X}) = \beta^{\dagger}(\underline{X}),$$
$$\text{(iii)} \quad \alpha|(\underline{X}) = \alpha|^{\dagger}(\underline{X}), \qquad \beta|(\underline{X}) = \beta|^{\dagger}(\underline{X}),$$
$$\text{(iv)} \quad \alpha(\underline{X}) + \beta(\underline{X}) = 1. \tag{4}$$

Especially when $\underline{X}|_{S^{2n-1}} = \underline{W}$, then

$$\text{(v)} \quad \alpha^2(\underline{W}) = \alpha(\underline{W}), \qquad \beta^2(\underline{W}) = \beta|(\underline{W}), \tag{5}$$
$$\text{(vi)} \quad \alpha|^2(\underline{W}) = \alpha|(\underline{W}), \qquad \beta|^2(\underline{W}) = \beta(\underline{W}). \tag{6}$$

Related results can be also found in [15,16] or monographs on Fourier analysis elsewhere.

In what follows we introduce matrix functions

$$\underline{\alpha} = \frac{1}{2}\begin{pmatrix} \alpha + \alpha| & -\alpha + \alpha| \\ -\alpha + \alpha| & \alpha + \alpha| \end{pmatrix}, \qquad \underline{\beta} = \frac{1}{2}\begin{pmatrix} \beta + \beta| & -\beta + \beta| \\ -\beta + \beta| & \beta + \beta| \end{pmatrix},$$

where α, β, $\alpha|$, $\beta|$ are for short of $\alpha(\underline{X})$, $\beta(\underline{X})$, $\alpha|(\underline{X})$, $\beta|(\underline{X})$. In the following context when without confusion and ambiguity, the independent variables of considered functions are omitted.

Making use of the above Lemma 3.1 and directly calculating of the matrix functions, we get the following lemma.

Lemma 3.2. *Suppose the functions $\underline{\alpha}$ and $\underline{\beta}$ as above. Then*

$$\text{(i)} \quad \underline{\alpha}\underline{\beta} = \frac{1 - |\underline{X}|^2}{4}\mathbf{1}, \tag{7}$$
$$\text{(ii)} \quad \underline{\alpha} = \underline{\alpha}^{\dagger}, \qquad \underline{\beta} = \underline{\beta}^{\dagger},$$
$$\text{(iii)} \quad \underline{\alpha} + \underline{\beta} = \mathbf{1}, \tag{8}$$

where $\mathbf{1}$ denotes (2×2) identity matrix.

Particularly, when $\underline{X}|_{S^{2n-1}} = \underline{W}$, then

$$\underline{\alpha}^2 = \underline{\alpha}, \qquad \underline{\beta}^2 = \underline{\beta}, \qquad \underline{\alpha}\underline{\beta} = \mathbf{0}. \tag{9}$$

The half Dirichlet problems with respect to the matrix functions $\underline{\alpha}$ and $\underline{\beta}$ are formulated as follows.

Problem I. Given the boundary data $\mathcal{G}_2^1 \in \mathbf{L}_p(\partial\Omega, \mathbb{C}_{2n})$, find the function $\mathcal{L}_2^1$ such that

$$\text{(i)} \quad \begin{cases} \mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X}) = \mathbf{0}, & \underline{X} \in B(1), \\ \underline{\alpha}\mathcal{L}_2^1(\underline{W}) = \underline{\alpha}\mathcal{G}_2^1(\underline{W}), & \underline{W} \in S^{2n-1}, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X}) = \mathbf{0}, & \underline{X} \in B(1), \\ \underline{\beta}\mathcal{L}_2^1(\underline{W}) = \underline{\beta}\mathcal{G}_2^1(\underline{W}), & \underline{W} \in S^{2n-1}, \end{cases}$$

where the matrix function $\mathcal{G}_2^1 = \left(\begin{smallmatrix} G_1 & G_2 \\ G_2 & G_1 \end{smallmatrix}\right)$ is defined similarly to $\mathcal{L}_2^1$.

Theorem 3.1. *For the above half Dirichlet problems* (i) *and* (ii) *there exist respectively the unique solutions. Moreover, the solutions to* (i) *and to* (ii) *are given respectively as follows*

$$\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} \triangleq \left[\mathbf{C}2\underline{\alpha}\mathcal{G}_2^1\right](\underline{X}) = \int_{S^{2n-1}} \mathcal{E}(\underline{V} - \underline{Z})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, 2\underline{\alpha}\widetilde{\mathcal{G}}_2^1(\underline{Y}), \quad \underline{X} \in \overline{B}(1), \tag{10}$$

$$\mathcal{L}_2^1(\underline{X})_{\underline{\beta}} \triangleq \left[\mathbf{C}2\underline{\beta}\mathcal{G}_2^1\right](\underline{X}) = \int_{S^{2n-1}} \mathcal{E}(\underline{V} - \underline{Z})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, 2\underline{\beta}\widetilde{\mathcal{G}}_2^1(\underline{Y}), \quad \underline{X} \in \overline{B}(1), \tag{11}$$

where $\widetilde{\mathcal{G}}_2^1 = (-1)^{\frac{n(n+1)}{2}}(2i)^{-n}\mathcal{G}_2^1$ *and the referred singular integrals in the terms of* (10) *and* (11), *called the Hermitian Cauchy transformations of matrix functions* $2\underline{\alpha}\widetilde{\mathcal{G}}_2^1$ *and* $2\underline{\beta}\widetilde{\mathcal{G}}_2^1$, *exist in the sense of Cauchy principal value as above.*

Proof. When $\underline{X} \in B(1)$, by means of (i) in Lemma 2.1, we get $\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} = \mathbf{0}$ and $\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}} = \mathbf{0}$. It is sufficient to consider the problem (i). Similarly to the problem (ii). In the following we will prove that $\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}}$ satisfies the corresponding boundary condition for arbitrary $\underline{W} \in S^{2n-1}$. Write $\underline{X} = r\underline{W}$, $\underline{W} \in S^{2n-1}$ $(0 < r < 1)$. Let

$$\mathcal{L}_2^1(\underline{W}) = \lim_{r\to 1^-} \mathcal{L}_2^1(r\underline{W})_{\underline{\alpha}}, \quad \underline{W} \in S^{2n-1}.$$

According to (iii) in Lemma 2.1, $\mathcal{L}_2^1(\underline{W})$ belongs to $\mathbf{L}_p(S^{2n-1}, \mathbb{C}_{2n})$. So $\underline{\alpha}\mathcal{L}_2^1(\underline{W})$ belongs to $\mathbf{L}_p(S^{2n-1}, \mathbb{C}_{2n})$. By means of (ii) in Lemma 2.1, we get

$$\mathcal{L}_2^1(\underline{W}) = \underline{\alpha}\mathcal{G}_2^1(\underline{W}) + \left[\mathbf{H}\underline{\alpha}\mathcal{G}_2^1\right](\underline{W}).$$

Then making use of Lemma 3.2, we have

$$\lim_{r\to 1^-} \underline{\alpha}\mathcal{L}_2^1(r\underline{W})_{\underline{\alpha}} = \underline{\alpha}^2\mathcal{G}_2^1(\underline{W}) + \underline{\alpha}\left[\mathbf{H}\underline{\alpha}\mathcal{G}_2^1\right](\underline{W}) = \underline{\alpha}\mathcal{G}_2^1(\underline{W}) + \underline{\alpha}\left[\mathbf{H}\underline{\alpha}\mathcal{G}_2^1\right](\underline{W}), \tag{12}$$

where

$$\begin{aligned} &\underline{\alpha}\left[\mathbf{H}\underline{\alpha}\mathcal{G}_2^1\right](\underline{W}) \\ &\quad = \frac{1}{2}\begin{pmatrix} \alpha\mathcal{H}\alpha G_1 + \alpha|\mathcal{H}|\alpha|G_1 + \alpha|\mathcal{H}|\alpha|G_2 - \alpha\mathcal{H}\alpha G_2 & \alpha\mathcal{H}\alpha G_2 + \alpha|\mathcal{H}|\alpha|G_2 + \alpha|\mathcal{H}|\alpha|G_1 - \alpha\mathcal{H}\alpha G_1 \\ \alpha|\mathcal{H}|\alpha|G_1 - \alpha\mathcal{H}\alpha G_1 + \alpha\mathcal{H}\alpha G_2 + \alpha|\mathcal{H}|\alpha|G_2 & \alpha|\mathcal{H}|\alpha|G_2 - \alpha\mathcal{H}\alpha G_2 + \alpha\mathcal{H}\alpha G_1 + \alpha|\mathcal{H}|\alpha|G_1 \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha\mathcal{H}\alpha G_i &= \text{p.v.}2(1 + i\underline{W}) \int_{S^{2n-1}} E(\underline{\zeta} - \underline{W})\underline{\zeta}(1 + i\underline{\zeta})G_i(\underline{\zeta})\, dS_{\underline{\zeta}} \\ &= \text{p.v.}\frac{2}{w_{2n}}(1 + i\underline{W}) \int_{S^{2n-1}} \frac{\underline{W} - \underline{\zeta}}{|\underline{W} - \underline{\zeta}|^{2n}}\underline{\zeta}(1 + i\underline{\zeta})G_i(\underline{\zeta})\, dS_{\underline{\zeta}} \quad (i = 1, 2), \\ \alpha|\mathcal{H}|\alpha|G_i &= \text{p.v.}2(1 + i\underline{W}|) \int_{S^{2n-1}} E(\underline{\zeta}| - \underline{W}|)\underline{\zeta}(1 + i\underline{\zeta}|)G_i(\underline{\zeta})\, dS_{\underline{\zeta}} \\ &= \text{p.v.}\frac{2}{w_{2n}}(1 + i\underline{W}|) \int_{S^{2n-1}} \frac{\underline{W}| - \underline{\zeta}|}{|\underline{W} - \underline{\zeta}|^{2n}}\underline{\zeta}|(1 + i\underline{\zeta}|)G_i(\underline{\zeta})\, dS_{\underline{\zeta}} \quad (i = 1, 2). \end{aligned}$$

Since $\underline{W}, \underline{\zeta} \in S^{2n-1}$, then $\underline{W}|, \underline{\zeta}| \in S^{2n-1}$. Therefore, we have

$$(1+i\underline{W})(\underline{W}-\underline{\zeta})\underline{\zeta}(1+i\underline{\zeta})=0,$$
$$(1+i\underline{W}|)(\underline{W}|-\underline{\zeta}|)\underline{\zeta}|(1+i\underline{\zeta}|)=0.$$

Hence

$$\begin{aligned}&\underline{\alpha}\big[\mathbf{H}\underline{\alpha}\mathcal{G}_2^1\big](\underline{W})=\mathbf{0}, \quad \underline{W}\in S^{2n-1},\\ &\text{i.e.} \quad \lim_{r\to 1^-}\underline{\alpha}\mathcal{L}_2^1(r\underline{W})_{\underline{\alpha}}=\underline{\alpha}\mathcal{G}_2^1(\underline{W}), \quad \underline{W}\in S^{2n-1}.\end{aligned} \tag{13}$$

Suppose that $\mathcal{U}_2^1$ and $\mathcal{V}_2^1$ are both solutions of the problem (i), where the matrix functions $\mathcal{U}_2^1$ and $\mathcal{V}_2^1$ are defined similarly to $\mathcal{L}_2^1$. Putting $\mathcal{T}_2^1=\mathcal{U}_2^1-\mathcal{V}_2^1$, we have

$$\begin{cases}\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{T}_2^1(\underline{X})=\mathbf{0}, & \underline{X}\in B(1),\\ \underline{\alpha}\mathcal{T}_2^1(\underline{W})=\mathbf{0}, & \underline{W}\in S^{2n-1}.\end{cases} \tag{$*$}$$

Since

$$\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}=\begin{pmatrix}\partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger}\\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}}\end{pmatrix} \quad \text{with } 4(\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)})(\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)})^\dagger=\begin{pmatrix}\Delta_{2n} & 0\\ 0 & \Delta_{2n}\end{pmatrix}\triangleq\underline{\Delta},$$

then

$$\begin{cases}\partial_{\underline{Z}^\dagger}T_1+\partial_{\underline{Z}}T_2=0, & \underline{X}\in B(1),\\ \partial_{\underline{Z}^\dagger}T_2+\partial_{\underline{Z}}T_1=0, & \underline{X}\in B(1),\\ \alpha(T_1-T_2)=0, & \underline{W}\in S^{2n-1},\\ \alpha|(T_1+T_2)=0, & \underline{W}\in S^{2n-1}.\end{cases}$$

As $\partial_{\underline{Z}^\dagger}=\frac{1}{4}(\partial_{\underline{X}}+i\partial_{\underline{X}|})$, $\partial_{\underline{Z}}=-\frac{1}{4}(\partial_{\underline{X}}-i\partial_{\underline{X}|})$, $\Delta_{2n}=-\partial_{\underline{X}}^2=-\partial_{\underline{X}|}^2$, then we get

$$\begin{cases}\partial_{\underline{X}}(T_1-T_2)=0, & \underline{X}\in B(1),\\ \partial_{\underline{X}|}(T_1+T_2)=0, & \underline{X}\in B(1),\end{cases} \quad \text{and} \quad \begin{cases}\Delta_{2n}(T_1-T_2)=0, & \underline{X}\in B(1),\\ \Delta_{2n}(T_1+T_2)=0, & \underline{X}\in B(1).\end{cases}$$

Therefore, by directly calculating, we have

$$\Delta_{2n}\big[\underline{X}(T_1-T_2)\big]=0, \quad \underline{X}\in B(1),$$
$$\Delta_{2n}\big[\underline{X}|(T_1+T_2)\big]=0, \quad \underline{X}\in B(1).$$

Hence we get

$$\begin{cases}\Delta_{2n}\big[\alpha(T_1-T_2)\big]=0, & \underline{X}\in B(1), \\ \alpha(T_1-T_2)=0, & \underline{W}\in S^{2n-1}, \quad \text{(iii)}\\ \Delta_{2n}\big[\alpha|(T_1+T_2)\big]=0, & \underline{X}\in B(1), \\ \alpha|(T_1+T_2)=0, & \underline{W}\in S^{2n-1}. \quad \text{(iv)}\end{cases}$$

Associating the maximum principle for classical harmonic functions (see [24,25]) on $\overline{B}(1)$ with (iii) and (iv) respectively, we have

$$\begin{cases}\alpha\big(T_1(\underline{X})-T_2(\underline{X})\big)\equiv 0, & \underline{X}\in B(1),\\ \alpha|\big(T_1(\underline{X})+T_2(\underline{X})\big)\equiv 0, & \underline{X}\in B(1).\end{cases}$$

Making use of the term (3), we get

$$\begin{cases}\beta\alpha\big(T_1(\underline{X})-T_2(\underline{X})\big)=\dfrac{1-|\underline{X}|^2}{4}\big(T_1(\underline{X})-T_2(\underline{X})\big)\equiv 0, & \underline{X}\in B(1),\\ \beta|\alpha|\big(T_1(\underline{X})+T_2(\underline{X})\big)=\dfrac{1-|\underline{X}|^2}{4}\big(T_1(\underline{X})+T_2(\underline{X})\big)\equiv 0, & \underline{X}\in B(1).\end{cases}$$

Therefore

$$\begin{cases}T_1(\underline{X})-T_2(\underline{X})\equiv 0, & \underline{X}\in B(1),\\ T_1(\underline{X})+T_2(\underline{X})\equiv 0, & \underline{X}\in B(1),\end{cases}$$

i.e.

$$T_1(\underline{X}) \equiv 0, \qquad T_2(\underline{X}) \equiv 0, \quad \underline{X} \in B(1).$$

So the solution of the problem (i) is unique. The proof of the result is completed. □

As a corollary, we get the solution to the Dirichlet problem for circulant (2×2) matrix functions.

Corollary 3.1 *(Dirichlet problem). Given the boundary data $\mathcal{G}_2^1 \in \mathbf{L}_p(S^{2n-1}, \mathbb{C}_{2n})$, find the function $\mathcal{K}_2^1$ such that*

$$\text{(v)} \quad \begin{cases} \underline{\Delta}\mathcal{K}_2^1(\underline{X}) = \mathbf{0}, & \underline{X} \in B(1), \\ \mathcal{K}_2^1(\underline{W}) = \mathcal{G}_2^1(\underline{W}), & \underline{W} \in S^{2n-1} \end{cases} \quad \Leftrightarrow \quad \text{(vi)} \quad \begin{cases} \Delta_{2n}K_1(\underline{X}) = 0, & \underline{X} \in B(1), \\ K_1(\underline{W}) = G_1(\underline{W}), & \underline{W} \in S^{2n-1}, \\ \Delta_{2n}K_2(\underline{X}) = 0, & \underline{X} \in B(1), \\ K_2(\underline{W}) = G_2(\underline{W}), & \underline{W} \in S^{2n-1}, \end{cases}$$

where $\mathcal{G}_2^1$ as above and $\mathcal{K}_2^1 = \left(\begin{smallmatrix} K_1 & K_2 \\ K_2 & K_1 \end{smallmatrix}\right)$ is defined similarly to $\mathcal{L}_2^1$.

Then to the above Dirichlet problem (v) *there exists a unique solution. Moreover, the solution is given by*

$$\mathcal{K}_2^1(\underline{X}) = \underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} + \underline{\beta}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}}, \quad \underline{X} \in \overline{B}(1), \tag{14}$$

where $\underline{\alpha}$, $\underline{\beta}$, $\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}}$, $\mathcal{L}_2^1(\underline{X})_{\underline{\beta}}$ as above.

Proof. Since for arbitrary $\underline{X} \in \overline{B}(1)$,

$$\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} \triangleq \left[\mathbf{C}2\underline{\alpha}\mathcal{G}_2^1\right](\underline{X}) = \int\limits_{S^{2n-1}} \mathcal{E}(\underline{V} - \underline{Z})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, 2\underline{\alpha}\widetilde{\mathcal{G}}_2^1(\underline{Y}), \tag{15}$$

$$\mathcal{L}_2^1(\underline{X})_{\underline{\beta}} \triangleq \left[\mathbf{C}2\underline{\beta}\mathcal{G}_2^1\right](\underline{X}) = \int\limits_{S^{2n-1}} \mathcal{E}(\underline{V} - \underline{Z})\, d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\, 2\underline{\beta}\widetilde{\mathcal{G}}_2^1(\underline{Y}), \tag{16}$$

where $\widetilde{\mathcal{G}}_2^1 = (-1)^{\frac{n(n+1)}{2}}(2i)^{-n}\mathcal{G}_2^1 = \left(\begin{smallmatrix} \widetilde{G}_1 & \widetilde{G}_2 \\ \widetilde{G}_2 & \widetilde{G}_1 \end{smallmatrix}\right)$.

As $\partial_{\underline{X}}(\mathcal{C}[\widetilde{G}_i](\underline{X})) = 0$, $\partial_{\underline{X}|}(\mathcal{C}|[\widetilde{G}_i](\underline{X}|)) = 0$ $(i = 1, 2)$, then $\mathcal{C}[\widetilde{G}_i](\underline{X})$, $\mathcal{C}|[\widetilde{G}_i](\underline{X}|)$, $\underline{X}(\mathcal{C}[\widetilde{G}_i](\underline{X}))$ and $\underline{X}|(\mathcal{C}|[\widetilde{G}_i](\underline{X}|))$ $(i = 1, 2)$ are harmonic (see for instance [5] or by direct calculation) where $\mathcal{C}[\widetilde{G}_i](\underline{X})$, $\mathcal{C}|[\widetilde{G}_i](\underline{X}|)$ are similarly defined to $\mathcal{C}[L_i](\underline{Y})$, $\mathcal{C}|[L_i](\underline{Y}|)$ as above, and are denoted by $\mathcal{C}[\widetilde{G}_i]$, $\mathcal{C}|[\widetilde{G}_i]$, for convenience, without confusion in the following.

Associating the orthogonal Cauchy transformations with the Hermitian Cauchy transformations (1) and (2), we get

$$\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} = \frac{1}{2}\left[\begin{pmatrix} \mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & -\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \\ -\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & \mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \end{pmatrix} + \begin{pmatrix} \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \\ \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \end{pmatrix}\right].$$

Since

$$\underline{\alpha}\begin{pmatrix} \mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & -\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \\ -\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & \mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \end{pmatrix} = \begin{pmatrix} \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \\ -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \end{pmatrix},$$

and

$$\underline{\alpha}\begin{pmatrix} \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \\ \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \end{pmatrix} = \begin{pmatrix} \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \\ \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \end{pmatrix},$$

then

$$\underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} = \frac{1}{2}\left[\begin{pmatrix} \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \\ -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \end{pmatrix} + \begin{pmatrix} \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \\ \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \end{pmatrix}\right].$$

Therefore, we obtain

$$\underline{\Delta}\left[\begin{pmatrix} \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \\ -\alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] & \alpha\mathcal{C}[\widetilde{G}_1 - \widetilde{G}_2] \end{pmatrix} + \begin{pmatrix} \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \\ \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] & \alpha|\mathcal{C}|[\widetilde{G}_1 + \widetilde{G}_2] \end{pmatrix}\right] = \mathbf{0},$$

i.e.

$$\underline{\Delta}\left[\underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}}\right] = \mathbf{0}.$$

Similarly,

$$\underline{\Delta}\big[\underline{\beta}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}}\big]=\mathbf{0}.$$

So we get

$$\underline{\Delta}\mathcal{K}_2^1(\underline{X})=\underline{\Delta}\big[\underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}}+\underline{\beta}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}}\big]=\mathbf{0},\quad \underline{X}\in B(1).$$

Moreover, associating the term of (13) with the term of (8), for arbitrary $\underline{W}\in S^{2n-1}$, we have

$$\mathcal{K}_2^1(\underline{W})=\lim_{r\to 1^-}\mathcal{K}_2^1(r\underline{W})=\lim_{r\to 1^-}\big(\underline{\alpha}\mathcal{L}_2^1(r\underline{W})_{\underline{\alpha}}+\underline{\beta}\mathcal{L}_2^1(r\underline{W})_{\underline{\beta}}\big)=\underline{\alpha}\mathcal{G}_2^1(\underline{W})+\underline{\beta}\mathcal{G}_2^1(\underline{W})=\mathcal{G}_2^1(\underline{W}).$$

The proof of the result is completed. □

Remark 3.1. The solutions to the above half Dirichlet problems (i) and (ii) respectively give rise to the solution to the Dirichlet problem (v) for circulant (2×2) matrix functions. That is, the solution to the Dirichlet problem for circulant (2×2) matrix functions is given by the way of Hermitian Cauchy transformation merely.

Problem II. Given the boundary data $\mathcal{G}_0\in\mathbf{L}_p(S^{2n-1},\mathbb{C}_{2n})$, find the function $\mathcal{L}_0$ such that

$$\text{(vii)}\quad \begin{cases}\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_0(\underline{X})=\mathbf{0}, & \underline{X}\in B(1),\\ \underline{\alpha}\mathcal{L}_0(\underline{W})=\underline{\alpha}\mathcal{G}_0(\underline{W}), & \underline{W}\in S^{2n-1},\end{cases}$$

$$\text{(viii)}\quad \begin{cases}\mathbf{D}_{(\underline{Z},\underline{Z}^\dagger)}\mathcal{L}_0(\underline{X})=\mathbf{0}, & \underline{X}\in B(1),\\ \underline{\beta}\mathcal{L}_0(\underline{W})=\underline{\beta}\mathcal{G}_0(\underline{W}), & \underline{W}\in S^{2n-1},\end{cases}$$

where $\mathcal{G}_0=\left(\begin{smallmatrix}G&0\\0&G\end{smallmatrix}\right)$ is defined similarly to $\mathcal{L}_0$.

Remark 3.2. Let $\mathbb{S}=\mathbb{C}_{2n}I\approxeq\mathbb{C}_nI$ denote the spinor space and $\mathbb{S}=\bigoplus_{j=1}^n\mathbb{S}_j=\bigoplus_{j=1}^n(\mathbb{C}\wedge_n^\dagger)^{(j)}I$, where $(\mathbb{C}\wedge_n^\dagger)^{(j)}$ is the Grassmann algebra generated by the Witt basis elements $\{f_1^\dagger,f_2^\dagger,\ldots,f_n^\dagger\}$ and $\mathbb{S}_j=(\mathbb{C}\wedge_n^\dagger)^{(j)}I$ $(j=0,1,2,\ldots,n)$ consists of j-vectors from $(\mathbb{C}\wedge_n^\dagger)^{(j)}$ multiplied by the primitive idempotent I as in Section 2 (see references e.g. [6–8,10]). When L, G as above only take values in the homogeneous n-space of spinor space $\mathbb{S}_n$, i.e.

$$L(z_1,z_2,\ldots,z_n)=l_n(z_1,z_2,\ldots,z_n)f_1^\dagger f_2^\dagger\cdots f_n^\dagger I,$$
$$G(z_1,z_2,\ldots,z_n)=g_n(z_1,z_2,\ldots,z_n)f_1^\dagger f_2^\dagger\cdots f_n^\dagger I,$$

where l_n, g_n are both complex-valued functions defined in $\mathbb{R}^{2n}\cong\mathbb{C}^n$, $\partial_{z_j^c}$ as in Section 2, then

$$\begin{pmatrix}\partial_{\underline{Z}}L & \partial_{\underline{Z}^\dagger}L\\ \partial_{\underline{Z}^\dagger}L & \partial_{\underline{Z}}L\end{pmatrix}=\begin{pmatrix}\sum_{j=1}^n\partial_{z_j}l_nf_j^\dagger f_1^\dagger f_2^\dagger\cdots f_n^\dagger I & \sum_{j=1}^n\partial_{z_j^c}l_nf_jf_1^\dagger f_2^\dagger\cdots f_n^\dagger I\\ \sum_{j=1}^n\partial_{z_j^c}l_nf_jf_1^\dagger f_2^\dagger\cdots f_n^\dagger I & \sum_{j=1}^n\partial_{z_j}l_nf_j^\dagger f_1^\dagger f_2^\dagger\cdots f_n^\dagger I\end{pmatrix}=\mathbf{0}.$$

Making use of the property of Witt basis, we have

$$\partial_{z_j^c}l_n=0\quad (j=1,2,\ldots,n).$$

Associating (vii) and (viii) with the above term, we get

$$\begin{cases}\partial_{z_j^c}l_n=0 & (j=1,2,\ldots,n),\ (z_1,z_2,\ldots,z_n)\in B(1)\subset\mathbb{C}^n,\\ l_n=g_n & (z_1,z_2,\ldots,z_n)\in S^{2n-1}\subset\mathbb{C}^n,\end{cases}$$

which exactly corresponds to the classical Dirichlet condition for holomorphic functions of several complex variables. Further, when $j=1$ is considered, it reduces to the case as follows

$$\begin{cases}\partial_{z_1^c}l_n=0, & z_1\in b(1)=\{z_1\in\mathbb{C}\mid |z_1|<1\},\\ l_n=g_n, & z_1\in\partial b=\{z_1\in\mathbb{C}\mid |z_1|=1\}.\end{cases}$$

This implies that the classical Dirichlet problems for analytic functions of one complex variable and holomorphic functions of several complex variables (see e.g. [26,23]) can be solved by means of Hermitian Clifford analysis.

As the special cases of Theorem 3.1, we directly give the solutions to the above Dirichlet problems (vii) and (viii).

Theorem 3.2. *For the Dirichlet problems* (vii) *and* (viii) *there exist the unique solutions respectively. Moreover, the solutions are respectively given as follows*

$$\mathcal{L}_0(\underline{X})_{\underline{\alpha}} \triangleq [\mathbf{C}2\underline{\alpha}\mathcal{G}_0](\underline{X}) = \int_{S^{2n-1}} \mathcal{E}(\underline{V}-\underline{Z})\,d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\,2\underline{\alpha}\widetilde{\mathcal{G}}_0(\underline{Y}), \quad \underline{X}\in\overline{B}(1), \tag{17}$$

$$\mathcal{L}_0(\underline{X})_{\underline{\beta}} \triangleq [\mathbf{C}2\underline{\beta}\mathcal{G}_0](\underline{X}) = \int_{S^{2n-1}} \mathcal{E}(\underline{V}-\underline{Z})\,d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}\,2\underline{\beta}\widetilde{\mathcal{G}}_0(\underline{Y}), \quad \underline{X}\in\overline{B}(1), \tag{18}$$

where $\widetilde{\mathcal{G}}_0 = (-1)^{\frac{n(n+1)}{2}}(2i)^{-n}\mathcal{G}_0$ *and the referred singular integrals in the terms of* (17) *and* (18)*, called the Hermitian Cauchy transformations of matrix functions* $2\underline{\alpha}\widetilde{\mathcal{G}}_0$ *and* $2\underline{\beta}\widetilde{\mathcal{G}}_0$*, exist in the sense of the Cauchy principal value as above.*

Corollary 3.2 *(Classical Dirichlet problem). Given the boundary data* $\mathcal{G}_0 \in \mathbf{L}_p(S^{2n-1}, \mathbb{C}_{2n})$ *as above, find the function* $\mathcal{K}_0$*, where* $\mathcal{K}_0$ *is defined similarly to* $\mathcal{L}_0$ *as above, such that*

$$\text{(ix)}\quad \begin{cases} \underline{\Delta}\mathcal{K}_0(\underline{X}) = \mathbf{0}, & \underline{X}\in B(1),\\ \mathcal{K}_0(\underline{W}) = \mathcal{G}_0(\underline{W}), & \underline{W}\in S^{2n-1} \end{cases} \quad\Leftrightarrow\quad \text{(x)}\quad \begin{cases} \Delta_{2n}K(\underline{X}) = 0, & \underline{X}\in B(1),\\ K(\underline{W}) = G(\underline{W}), & \underline{W}\in S^{2n-1}. \end{cases}$$

Then for the above classical Dirichlet problem (vii) *there exists the unique solution. Moreover, the solution is expressed by*

$$\mathcal{K}_0(\underline{X}) = \underline{\alpha}\mathcal{L}_0(\underline{X})_{\underline{\alpha}} + \underline{\beta}\mathcal{L}_0(\underline{X})_{\underline{\beta}}, \quad \underline{X}\in\overline{B}(1), \tag{19}$$

where $\underline{\alpha}$, $\underline{\beta}$, $\mathcal{L}_0(\underline{X})_{\underline{\alpha}}$, $\mathcal{L}_0(\underline{X})_{\underline{\beta}}$, $\mathcal{G}_0$ *as above and* $\mathcal{K}_0 = \left(\begin{smallmatrix} K & 0\\ 0 & K\end{smallmatrix}\right)$ *is defined similarly to* $\mathcal{L}_0$.

Proof. As the special cases of Corollary 3.1, we directly give the solutions to the Dirichlet problems (ix) and (x) as above. □

Remark 3.3. The above Corollary 3.2 implies that in even dimensional Euclidean space the solution to the classical Dirichlet problem can be given by the way of the Hermitian Cauchy transformation merely.

Remark 3.4. All results on the above half Dirichlet problems and Dirichlet problems are given when the boundary data belong to $\mathbf{L}_p(S^{2n-1})$ $(1 < p < +\infty)$ and the considered functions take non-tangential boundary values by the way of Hermitian Clifford analysis. Moreover, the same problems may be considered and the analogous results can be obtained while the boundary data belong to $\mathbb{H}^\mu(S^{2n-1}, \mathbb{C}_{2n})$ $(0 < \mu \leqslant 1)$ in the setting of Hermitian Clifford analysis.

4. Decomposition theorems

In this section we will give the decomposition theorem of the Poisson kernel of matrix Laplace operator and further derive the decomposition theorems of solution space to harmonic functions in the framework of circulant (2×2) matrix form.

To begin with, we introduce the following function

$$\mathcal{P}(\underline{X}, \underline{W}) = \begin{pmatrix} P(\underline{X},\underline{\xi}) & 0\\ 0 & P(\underline{X},\underline{\xi})\end{pmatrix},$$

where

$$P(\underline{X},\underline{\xi}) = \frac{1}{w_{2n}}\frac{1-|\underline{X}|^2}{|\underline{X}-\underline{\xi}|^{2n}}, \quad \underline{X}\neq\underline{\xi}.$$

It is easy to check that $\underline{\Delta}\mathcal{P}(\underline{X},\underline{\xi}) = \mathbf{0}$. Hence $\mathcal{P}(\underline{X},\underline{\xi})$ is the kernel function of the matrix Laplace operator $\underline{\Delta}$ as same as the foregoing sections. Moreover, the solution to the above Dirichlet problem (v) could be also expressed in the term (see reference e.g. [24]) as follows

$$\mathcal{K}_2^1(\underline{X}) = \int_{S^{2n-1}} \mathcal{P}(\underline{X},\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\,dS_{\underline{\xi}}, \quad \underline{X}\in\overline{B}(1). \tag{20}$$

On the other hand, by directly calculating, we get

$$2\underline{\alpha}\mathcal{E}\,d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}\underline{\alpha} = \frac{1}{8}\begin{pmatrix} \alpha+\alpha| & -\alpha+\alpha|\\ -\alpha+\alpha| & \alpha+\alpha|\end{pmatrix}\begin{pmatrix} \varepsilon & \varepsilon^\dagger\\ \varepsilon^\dagger & \varepsilon\end{pmatrix}\begin{pmatrix} d\sigma_{\underline{\xi}} & -d\sigma_{\underline{\xi}^\dagger}\\ -d\sigma_{\underline{\xi}^\dagger} & d\sigma_{\underline{\xi}}\end{pmatrix}\begin{pmatrix} \alpha+\alpha| & -\alpha+\alpha|\\ -\alpha+\alpha| & \alpha+\alpha|\end{pmatrix},$$

where $\underline{\alpha}$, $\underline{\beta}$ as the foregoing sections and $\underline{\xi}$, $\underline{\xi}^\dagger$, $d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}$, $d\sigma_{\underline{\xi}}$ and $d\sigma_{\underline{\xi}^\dagger}$ are defined similarly to $\underline{Z}$, $\underline{Z}^\dagger$, $d\Sigma_{(\underline{Z},\underline{Z}^\dagger)}$, $d\sigma_{\underline{Z}}$ and $d\sigma_{\underline{Z}^\dagger}$.

Hence we obtain

$$2\underline{\alpha}\mathcal{E}\,d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}\underline{\alpha} = \begin{pmatrix} \alpha E\,d\sigma_{\underline{\xi}}\alpha + \alpha|E|d\sigma_{\underline{\xi}}|\alpha| & -\alpha E\,d\sigma_{\underline{\xi}}\alpha + \alpha|E|\,d\sigma_{\underline{\xi}}\alpha| \\ -\alpha E\,d\sigma_{\underline{\xi}}\alpha + \alpha|E|\,d\sigma_{\underline{\xi}}\alpha| & \alpha E\,d\sigma_{\underline{\xi}}\alpha + \alpha|E|d\sigma_{\underline{\xi}}|\alpha| \end{pmatrix}.$$

Similarly,

$$2\underline{\beta}\mathcal{E}\,d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}\underline{\beta} = \begin{pmatrix} \beta E\,d\sigma_{\underline{\xi}}\beta + \beta|E|d\sigma_{\underline{\xi}}|\beta| & -\beta E\,d\sigma_{\underline{\xi}}\beta + \beta|E|\,d\sigma_{\underline{\xi}}\beta| \\ -\beta E\,d\sigma_{\underline{\xi}}\beta + \beta|E|\,d\sigma_{\underline{\xi}}\beta| & \beta E\,d\sigma_{\underline{\xi}}\beta + \beta|E|d\sigma_{\underline{\xi}}|\beta| \end{pmatrix}.$$

For arbitrary $\underline{X} \in \mathbb{R}^{2n}$ and $\underline{\xi} \in S^{2n-1}$, we have

$$\begin{aligned}
&\alpha(\underline{X})E(\underline{X}-\underline{\xi})\,d\sigma_{\underline{\xi}}\alpha(\underline{\xi}) = \alpha(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\alpha(\underline{\xi})\,dS_{\underline{\xi}},\\
&\alpha|(\underline{X})E|(\underline{X}-\underline{\xi})\,d\sigma_{\underline{\xi}}|\alpha|(\underline{\xi}) = \alpha|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\alpha|(\underline{\xi})\,dS_{\underline{\xi}},\\
&\beta(\underline{X})E(\underline{X}-\underline{\xi})\,d\sigma_{\underline{\xi}}\beta(\underline{\xi}) = \beta(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\beta(\underline{\xi})\,dS_{\underline{\xi}},\\
&\beta|(\underline{X})E|(\underline{X}-\underline{\xi})\,d\sigma_{\underline{\xi}}|\alpha|(\underline{\xi}) = \beta|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\beta|(\underline{\xi})\,dS_{\underline{\xi}},
\end{aligned}$$

and

$$\begin{aligned}
&|\underline{X}-\underline{\xi}|^{2n}\big[\alpha(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\alpha(\underline{\xi}) + \beta(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\beta(\underline{\xi})\big] = 1-|\underline{X}|^2,\\
&|\underline{X}-\underline{\xi}|^{2n}\big[\alpha|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\alpha|(\underline{\xi}) + \beta|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\beta|(\underline{\xi})\big] = 1-|\underline{X}|^2.
\end{aligned}$$

Let us introduce the following matrix functions

$$C_{\underline{X}}^{\underline{\alpha}}(\underline{\xi}) = \begin{pmatrix} C_{11}(\underline{X},\underline{\xi}) & C_{12}(\underline{X},\underline{\xi}) \\ C_{12}(\underline{X},\overline{\underline{\xi}}) & C_{11}(\underline{X},\overline{\underline{\xi}}) \end{pmatrix}, \qquad C_{\underline{X}}^{\underline{\beta}}(\underline{\xi}) = \begin{pmatrix} C_{21}(\underline{X},\underline{\xi}) & C_{22}(\underline{X},\underline{\xi}) \\ C_{22}(\underline{X},\overline{\underline{\xi}}) & C_{21}(\underline{X},\overline{\underline{\xi}}) \end{pmatrix},$$

where

$$\begin{aligned}
&C_{11}(\underline{X},\underline{\xi}) = \alpha(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\alpha(\underline{\xi}) + \alpha|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\alpha|(\underline{\xi}),\\
&C_{12}(\underline{X},\underline{\xi}) = -\alpha(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\alpha(\underline{\xi}) + \alpha|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\alpha|(\underline{\xi}),\\
&C_{21}(\underline{X},\underline{\xi}) = \beta(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\beta(\underline{\xi}) + \beta|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\beta|(\underline{\xi}),\\
&C_{22}(\underline{X},\underline{\xi}) = -\beta(\underline{X})E(\underline{X}-\underline{\xi})\underline{\xi}\beta(\underline{\xi}) + \beta|(\underline{X})E|(\underline{X}-\underline{\xi})\underline{\xi}|\beta|(\underline{\xi}).
\end{aligned}$$

Therefore, we get the decomposition:

Theorem 4.1. *For arbitrary $\underline{X} \in \mathbb{R}^{2n}$ and $\underline{\xi} \in S^{2n-1}$, we have*

$$C_{\underline{X}}^{\underline{\alpha}}(\underline{\xi}) + C_{\underline{X}}^{\underline{\beta}}(\underline{\xi}) = \begin{pmatrix} P(\underline{X},\underline{\xi}) & 0 \\ 0 & P(\underline{X},\underline{\xi}) \end{pmatrix} = \mathcal{P}(\underline{X},\underline{\xi}). \tag{$**$}$$

Further for arbitrary $\underline{X} \in B(1)$, we have

$$\underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} = \int_{S^{2n-1}} \underline{\alpha}(\underline{X})\mathcal{E}(\underline{Z}-\underline{\xi})\,d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}\,\underline{\alpha}2\widetilde{\mathcal{G}}_2^1(\underline{\xi}) = \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\alpha}}(\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\,dS_{\underline{\xi}}, \tag{21}$$

$$\underline{\beta}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}} = \int_{S^{2n-1}} \underline{\beta}(\underline{X})\mathcal{E}(\underline{Z}-\underline{\xi})\,d\Sigma_{(\underline{\xi},\underline{\xi}^\dagger)}\,\underline{\beta}2\widetilde{\mathcal{G}}_2^1(\underline{\xi}) = \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\beta}}(\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\,dS_{\underline{\xi}}. \tag{22}$$

Associating the terms (21), (22) with the term (20), for arbitrary $\underline{X} \in B(1)$, we have

$$\mathcal{K}_2^1(\underline{X}) = \underline{\alpha}\mathcal{L}_2^1(\underline{X})_{\underline{\alpha}} + \underline{\beta}\mathcal{L}_2^1(\underline{X})_{\underline{\beta}}. \tag{23}$$

Particularly with respect to the classical Dirichlet problem (ix), for arbitrary $\underline{X} \in B(1)$, we have

$$\mathcal{K}_0(\underline{X}) = \underline{\alpha}\mathcal{L}_0(\underline{X})_{\underline{\alpha}} + \underline{\beta}\mathcal{L}_0(\underline{X})_{\underline{\beta}}. \tag{24}$$

In the following, for $1 < p < +\infty$ and $\mathcal{I}_2^1$, $\mathcal{I}_0$ which are circulant (2×2) matrix functions defined similarly to $\mathcal{L}_2^1$, $\mathcal{L}_0$, we denote

$$\begin{aligned}
\mathbb{H}_p\big(B(1), \mathbb{C}_{2n}\big) = \big\{&\mathcal{I}_2^1 : B(1) \to \mathbb{C}_{2n} \;\big|\; \underline{\Delta}\mathcal{I}_2^1 = \mathbf{0} \text{ and } \mathcal{I}_2^1 \text{ admits non-tangential boundary} \\
&\text{value which belongs to } \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{M}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big) = \big\{&\mathbf{C}\big[\underline{\alpha}\mathcal{I}_2^1\big] : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{I}_2^1 \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{M}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big) = \big\{&\mathbf{C}\big[\underline{\beta}\mathcal{I}_2^1\big] : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{I}_2^1 \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{H}_p\big(B(1), \mathbb{C}_{2n}\big)_0 = \big\{&\mathcal{I}_0 : B(1) \to \mathbb{C}_{2n} \;\big|\; \underline{\Delta}\mathcal{I}_0 = \mathbf{0} \text{ and } \mathcal{I}_0 \text{ admits non-tangential boundary} \\
&\text{value which belongs to } \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{M}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big)_0 = \big\{&\mathbf{C}[\underline{\alpha}\mathcal{I}_0] : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{I}_0 \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{M}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big)_0 = \big\{&\mathbf{C}[\underline{\beta}\mathcal{I}_0] : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{I}_0 \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{H}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big) = \big\{&\mathcal{P}^{\underline{\alpha}}\mathcal{G}_2^1(\underline{X}) : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{G}_2^1(\underline{X}) \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{H}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big)_0 = \big\{&\mathcal{P}^{\underline{\alpha}}\mathcal{G}_0(\underline{X}) : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{G}_0(\underline{X}) \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{H}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big) = \big\{&\mathcal{P}^{\underline{\beta}}\mathcal{G}_2^1(\underline{X}) : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{G}_2^1(\underline{X}) \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\}, \\
\mathbb{H}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big)_0 = \big\{&\mathcal{P}^{\underline{\beta}}\mathcal{G}_0(\underline{X}) : B(1) \to \mathbb{C}_{2n} \;\big|\; \mathcal{G}_0(\underline{X}) \in \mathbf{L}_p\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P}^{\underline{\alpha}}\mathcal{G}_2^1(\underline{X}) &= \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\alpha}}(\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\, dS_{\underline{\xi}}, \\
\mathcal{P}^{\underline{\alpha}}\mathcal{G}_0(\underline{X}) &= \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\alpha}}(\underline{\xi})\mathcal{G}_0(\underline{\xi})\, dS_{\underline{\xi}}, \\
\mathcal{P}^{\underline{\beta}}\mathcal{G}_2^1(\underline{X}) &= \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\beta}}(\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\, dS_{\underline{\xi}}, \\
\mathcal{P}^{\underline{\beta}}\mathcal{G}_0(\underline{X}) &= \int_{S^{2n-1}} C_{\underline{X}}^{\underline{\beta}}(\underline{\xi})\mathcal{G}_2^1(\underline{\xi})\, dS_{\underline{\xi}}.
\end{aligned}$$

It is easy to see that the term (20) implies the isomorphism between the space $\mathbb{H}_p(B(1), \mathbb{C}_{2n})$ and the space $\mathscr{L}_p(S^{2n-1}, \mathbb{C}_{2n}) = \{\mathcal{I}_2^1 \mid \mathcal{I}_2^1 \in \mathbf{L}_p(S^{2n-1}, \mathbb{C}_{2n})\}$.

Combining Corollary 3.1 with Corollary 3.2, we directly get the following theorem.

Theorem 4.2. *For $1 < p < +\infty$, we have*

$$\begin{aligned}
\text{(i)}\quad \mathbb{H}_p\big(B(1), \mathbb{C}_{2n}\big) &= \underline{\alpha}\mathbb{M}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big) + \underline{\beta}\mathbb{M}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big) \\
&= \mathbb{H}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big) + \mathbb{H}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big), \\
\text{(ii)}\quad \mathbb{H}_p\big(B(1), \mathbb{C}_{2n}\big)_0 &= \underline{\alpha}\mathbb{M}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big)_0 + \underline{\beta}\mathbb{M}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big)_0 \\
&= \mathbb{H}_p^{\underline{\alpha}}\big(B(1), \mathbb{C}_{2n}\big)_0 + \mathbb{H}_p^{\underline{\beta}}\big(B(1), \mathbb{C}_{2n}\big)_0.
\end{aligned}$$

In the following we introduce the vector space

$$\mathscr{L}_2\big(S^{2n-1}\big) = \left\{\mathcal{L}_2^1 = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix} \;\middle|\; L_1, L_2 \in \mathbf{L}_2\big(S^{2n-1}, \mathbb{C}_{2n}\big)\right\},$$

on which, inspired by $\mathbb{C}_{2n}$-valued inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{L}_2(S^{2n-1})$ given by

$$\langle L_1, L_2 \rangle = \int_{S^{2n-1}} L_1^{\dagger}(\underline{W}) L_2(\underline{W})\, dS_{\underline{W}},$$

we introduce the following bilinear form

$$\langle\cdot,\cdot\rangle_{\mathscr{L}_2} : \mathscr{L}_2\big(S^{2n-1}\big) \times \mathscr{L}_2\big(S^{2n-1}\big) \to (\mathbb{C}_{2n})^{2\times 2},$$
$$\left(\begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix}, \begin{pmatrix} K_1 & K_2 \\ K_2 & K_1 \end{pmatrix}\right) \mapsto \begin{pmatrix} \langle L_1, K_1\rangle + \langle L_2, K_2\rangle & \langle L_1, K_2\rangle + \langle L_2, K_1\rangle \\ \langle L_1, K_2\rangle + \langle L_2, K_1\rangle & \langle L_1, K_1\rangle + \langle L_2, K_2\rangle \end{pmatrix},$$

and we define the Hermitian conjugation on circulant elements of $(\mathbb{C}_{2n})^{2\times 2}$ as follows

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}^{\dagger} = \begin{pmatrix} a_1^{\dagger} & a_2^{\dagger} \\ a_2^{\dagger} & a_1^{\dagger} \end{pmatrix},$$

with $a_1, a_2 \in \mathbb{C}_{2n}$. Then $\langle\cdot,\cdot\rangle_{\mathscr{L}_2}$ is a $(\mathbb{C}_{2n})^{2\times 2}$-valued inner product (see [10]). Moreover, associating Lemma 3.2, we get the following decomposition

$$\mathscr{L}_2\big(S^{2n-1}\big) = \underline{\alpha}\mathscr{L}_2\big(S^{2n-1}\big) \oplus \underline{\beta}\mathscr{L}_2\big(S^{2n-1}\big).$$

Theorem 4.3. *For $p = 2$, we have*

(i) $\big(\mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big(\underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ \oplus \big(\underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+,$

(ii) $\big(\mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+ = \big(\underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+ \oplus \big(\underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+,$

(iii) $\mathscr{L}_2\big(S^{2n-1}, \mathbb{C}_{2n}\big) = \big(\mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ \oplus \big(\mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+,$

(iv) $\big(\mathscr{L}_2\big(S^{2n-1}, \mathbb{C}_{2n}\big)\big)_0 = \big(\mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)_0^+ \oplus \big(\mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)_0^+,$

where

$$\big(\mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big\{\textit{the boundary of } \mathcal{O}_2^1 \textit{ on } S^{2n-1} \;\big|\; \mathcal{O}_2^1 \in \mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)\big\},$$
$$\big(\underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big\{\textit{the boundary of } \mathcal{O}_2^1 \textit{ on } S^{2n-1} \;\big|\; \mathcal{O}_2^1 \in \underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big\},$$
$$\big(\underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big\{\textit{the boundary of } \mathcal{O}_2^1 \textit{ on } S^{2n-1} \;\big|\; \mathcal{O}_2^1 \in \underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big\},$$
$$\big(\mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+ = \big\{\textit{the boundary of } \mathcal{Q}_0 \textit{ on } S^{2n-1} \;\big|\; \mathcal{Q}_0 \in \mathbb{H}_2\big(B(1), \mathbb{C}_{2n}\big)_0\big\},$$
$$\big(\underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+ = \big\{\textit{the boundary of } \mathcal{Q}_0 \textit{ on } S^{2n-1} \;\big|\; \mathcal{Q}_0 \in \underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)_0\big\},$$
$$\big(\underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)_0\big)^+ = \big\{\textit{the boundary of } \mathcal{Q}_0 \textit{ on } S^{2n-1} \;\big|\; \mathcal{Q}_0 \in \underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)_0\big\},$$
$$\big(\mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big\{\textit{the boundary of } \mathcal{R}_2^1 \textit{ on } S^{2n-1} \;\big|\; \mathcal{R}_2^1 \in \mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big\},$$
$$\big(\mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ = \big\{\textit{the boundary of } \mathcal{R}_2^1 \textit{ on } S^{2n-1} \;\big|\; \mathcal{R}_2^1 \in \mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big\},$$
$$\big(\mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)_0^+ = \big\{\textit{the boundary of } \mathcal{R}_0 \textit{ on } S^{2n-1} \;\big|\; \mathcal{R}_0 \in \mathbb{H}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)_0\big\},$$
$$\big(\mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)_0^+ = \big\{\textit{the boundary of } \mathcal{R}_0 \textit{ on } S^{2n-1} \;\big|\; \mathcal{R}_0 \in \mathbb{H}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)_0\big\},$$

and the above spaces are well defined in the sense of the non-tangential boundary value.

Proof. Associating the term (21) with the term (22), making use of Lemma 2.1, it is sufficient to prove the cases (i) and (ii). Applying Theorem 4.2, we only need to prove the orthogonality of the above decompositions. The case (ii) is the special case of (i), it is sufficient for us to prove the case (i). For arbitrary

$$\mathcal{F}_2^1 = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix} \in \big(\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+ \quad \text{and} \quad \mathcal{M}_2^1 = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \in \big(\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+,$$

where $\mathcal{F}_2^1$, $\mathcal{M}_2^1$ are both defined similarly to $\mathcal{L}_2^1$, $\underline{W} \in S^{2n-1}$, then we have

$$\underline{\alpha}\mathcal{F}_2^1(\underline{W}) \in \big(\underline{\alpha}\mathbb{M}_2^{\alpha}\big(B(1), \mathbb{C}_{2n}\big)\big)^+, \qquad \underline{\beta}\mathcal{M}_2^1(\underline{W}) \in \big(\underline{\beta}\mathbb{M}_2^{\beta}\big(B(1), \mathbb{C}_{2n}\big)\big)^+,$$

and

$$\big(\underline{\alpha}\mathcal{F}_2^1, \underline{\beta}\mathcal{M}_2^1\big) = \left(\underline{\alpha}\begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix}, \underline{\beta}\begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}\right).$$

Since

$$\underline{\alpha}\mathcal{F}_2^1 = \frac{1}{2}\begin{pmatrix} \alpha(F_1 - F_2) + \alpha|(F_1 + F_2) & \alpha(F_2 - F_1) + \alpha|(F_1 + F_2) \\ \alpha(F_2 - F_1) + \alpha|(F_1 + F_2) & \alpha(F_1 - F_2) + \alpha|(F_1 + F_2) \end{pmatrix},$$

and

$$\underline{\beta}\mathcal{M}_2^1 = \frac{1}{2}\begin{pmatrix} \beta(M_1 - M_2) + \beta|(M_1 + M_2) & \beta(M_2 - M_1) + \beta|(M_1 + M_2) \\ \beta(M_2 - M_1) + \beta|(M_1 + M_2) & \beta(M_1 - M_2) + \beta|(M_1 + M_2) \end{pmatrix},$$

then making use of the above definition of the inner product $\langle \cdot, \cdot \rangle_{\mathscr{L}_2}$, we get

$$\left(\underline{\alpha}\mathcal{F}_2^1, \underline{\beta}\mathcal{M}_2^1\right) = \frac{1}{4}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} = {} & \big\langle \alpha(F_1 - F_2) + \alpha|(F_1 + F_2), \beta(M_1 - M_2) + \beta|(M_1 + M_2)\big\rangle \\ & + \big\langle \alpha(F_2 - F_1) + \alpha|(F_1 + F_2), \beta(M_2 - M_1) + \beta|(M_1 + M_2)\big\rangle, \end{aligned}$$

and

$$\begin{aligned} a_{12} = {} & \big\langle \alpha(F_1 - F_2) + \alpha|(F_1 + F_2), \beta(M_2 - M_1) + \beta|(M_1 + M_2)\big\rangle \\ & + \big\langle \alpha(F_2 - F_1) + \alpha|(F_1 + F_2), \beta(M_1 - M_2) + \beta|(M_1 + M_2)\big\rangle. \end{aligned}$$

Associating the terms of (8), (9) and applying the properties of $\langle \cdot, \cdot \rangle_{\mathbf{L}_2}$ in reference [10], we have

$$\left(\underline{\alpha}\mathcal{F}_2^1, \underline{\beta}\mathcal{M}_2^1\right) = \mathbf{0}.$$

The result follows. □

Acknowledgments

The authors cordially thank two anonymous referees for their valuable comments which lead to the improvement of this paper.

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